Section 3.7 Determinants

There are many matrices. We used some to solve 2-equation-2-unknown and 3-equation-3-unknowns systems.

Of all matrices, some are "square". That is they have the same number of rows as columns. For those matrices, there is a thing called a determinant, which is a number associated with that matrix.

Given this matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have its determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.
The value of the determinant is given by $ad - bc$.

The value of determinants for us is best illustrated by solving a system of equations. The person credited with this method was named Cramer.

Given: $\begin{cases} 6x - 2y = 4 \\ 7x + y = 13 \end{cases}$

Cramer computed 3 determinants. '

Det A is composed of the coefficients of x and y yielding: $\begin{vmatrix} 6 & -2 \\ 7 & 1 \end{vmatrix} = 6 + 14 = 20$

Det X is composed by ***replacing the column in Det A with the x values with the column containing the constant.s***
Det Y is composed by ***replacing the column in Det A with the y values with the column containing the constants.***

$Det\ X \begin{vmatrix} 4 & -2 \\ 13 & 1 \end{vmatrix} = 4 + 26 = 30$ $\quad$ $Det\ Y \begin{vmatrix} 6 & 4 \\ 7 & 13 \end{vmatrix} = 78 - 28 = 50$

Then Cramer defined $x = \dfrac{\text{Det } x}{\text{Det } A}$ $\quad y = \dfrac{\text{Det } y}{\text{Det } A}$

$x = \dfrac{30}{20}$ $\quad y = \dfrac{50}{20}$

Thus

$\left(\dfrac{3}{2}, \dfrac{5}{2}\right)$

Cramer's Rule is especially useful when the solutions are fractions because most of the fraction work is behind the scenes.

It is interesting to see why Cramer's rule works.

Consider a generic system of equations: $\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases} \rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$

I will multiply the top row by $-a_2$ and the bottom row by a_1 : giving me $\begin{bmatrix} -a_1a_2 & -a_2b_1 & -a_2c_1 \\ a_1a_2 & a_1b_2 & a_1c_2 \end{bmatrix}$

Now I add the top row to the bottom row:

$\begin{bmatrix} -a_1a_2 & -a_2b_1 & -a_2c_1 \\ 0 & a_1b_2-a_2b_1 & a_1c_2-a_2c_1 \end{bmatrix}$

extracting the equation from the bottom line of the matrix gives

$$\left(a_1b_2-a_2b_1\right)y=a_1c_2-a_2c_1$$

$$y=\frac{a_1c_2-a_2c_1}{\left(a_1b_2-a_2b_1\right)}$$

Cramer's rule would have the following solution:

Det A $=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1$

and

Det X $=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}=b_2c_1-b_1c_2$ Det Y $=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}=a_1c_2-a_2c_1$

$x=\frac{\text{Det X}}{\text{Det A}}=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}$ $y=\frac{\text{Det Y}}{\text{Det A}}=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$

Notice value for y in both versions.

This is the reason Cramer's Rule works.

3 X 3 systems of equations can be solved using Cramer's Rule as well.

For these you would find Det A, then $x=\frac{\text{det X}}{\text{det A}}$ $y=\frac{\text{det Y}}{\text{det A}}$ $z=\frac{\text{det Z}}{\text{det A}}$.

The tricky thing is to evaluate the 3X3 determinant.

There are several methods but we will look at only one method, the most basic one.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} \boxed{a_1} & \cancel{b_1} & \cancel{c_1} \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \cancel{a_1} & b_1 & c_1 \\ \boxed{a_2} & \cancel{b_2} & \cancel{c_2} \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \cancel{a_1} & b_1 & c_1 \\ \cancel{a_2} & b_2 & c_2 \\ \boxed{a_3} & \cancel{b_3} & \cancel{c_3} \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

This is called “expansion by minors”.

We start with row 1, column 1. I write that number down. Next I ignore that number’s row and its column. The remainder is a 2X2 determinant. On my version here I have drawn a line through the numbers I have ignored.

After that first item I move down to the next row, still in column 1. The unusual thing is that I must put a negative sign in before the a_2 value. I again ignore things in column 1 and now in row 2. Again, I have drawn a horizontal line through the numbers I am ignoring.

Finally, I have a plus the a_3 value and its 2X2 determinant.

An example (page 201)

$$\begin{cases} x-3y+7z=13 \\ x+y+z=1 \\ x-2y+3z=4 \end{cases}$$

$$\text{Det A} = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} - \begin{vmatrix} -3 & 7 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} -3 & 7 \\ 1 & 1 \end{vmatrix}$$

$$(3+2)-(-9+14)+(-3-7)$$

$$5-5-10$$

$$-10$$

$$\text{Det X} = \begin{vmatrix} 13 & -3 & 7 \\ 1 & 1 & 1 \\ 4 & -2 & 3 \end{vmatrix} = 13\begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} - \begin{vmatrix} -3 & 7 \\ -2 & 3 \end{vmatrix} + 4\begin{vmatrix} -3 & 7 \\ 1 & 1 \end{vmatrix}$$

$$= 13(3+2)-(-9+14)+4(-3-7)$$

$$= 65-5-40$$

$$= 20$$

$$\text{Det Y} = \begin{vmatrix} 1 & 13 & 7 \\ 1 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} - \begin{vmatrix} 13 & 7 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 13 & 7 \\ 1 & 1 \end{vmatrix}$$

$$= (3-4)-(39-28)+(13-7)$$

$$= -1-11+6$$

$$= -6$$

Fill in the missing data:

$$\text{Det Z} = \begin{vmatrix} & & 13 \\ & & 1 \\ & & 4 \end{vmatrix} = \begin{vmatrix} & \\ & \end{vmatrix} - \begin{vmatrix} & \\ & \end{vmatrix} + \begin{vmatrix} -3 & 13 \\ 1 & 1 \end{vmatrix}$$

$$= (4+)-(-12+26)+(-)$$

$$= 6-14-16$$

$$= -24$$

$$x = \frac{20}{-10} \qquad y = \frac{-6}{-10} \qquad z = \frac{-24}{-10}$$

$$x = -2 \qquad y = \frac{3}{5} \qquad z = \frac{12}{5}$$

$$\left(-2, \frac{3}{5}, \frac{12}{5}\right)$$

Section 4.3 Absolute Value Equations and Inequalities

The definition of absolute value: $|\bigstar| = \begin{cases} -\bigstar & \text{if } \bigstar < 0 \\ \bigstar & \text{if } \bigstar \geq 0 \end{cases}$

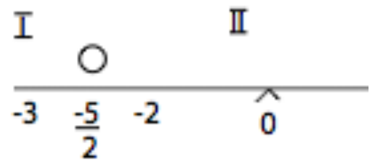
The first thing to notice that is different than the textbook is that we use stars in our definition rather than the x. The star represents "whatever is on the inside". The reason this is significant is that x could be negative but the inside could still be positive. (For example if $x + 4$ is on the inside, then when $x = -3$, the expression $x + 4$ is still positive.) We do not really care if x is negative or not. What concerns us is if the expression inside the absolute bars is negative or not.

Why do we care in the first place?

Absolute value, as we were introduced to it in chapter 1, is the distance from zero --- a non-negative quantity. Absolute values do not behave nicely in algebraic equations. The proper way to handle absolute values (especially when they are) in equations is to first remove them and then solve the resulting purely algebraic equations.

Example 2A page 243 done properly:

$|2x+5|=13$ We *always* draw a number line to illustrate the intervals.



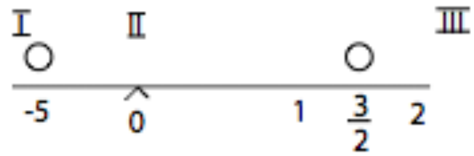
The two intervals are: $x<-\frac{5}{2}$ and $x\geq\frac{5}{2}$

If $x<-\frac{5}{2}$	or	If $x\geq-\frac{5}{2}$
$-(2x+5)=13$		$(2x+5)=13$
$-2x-5=13$		$2x+5=13$
$-2x=18$		$2x=8$
$x=-9$		$x=4$
is $-9<-\frac{5}{2}$		is $4\geq-\frac{5}{2}$
yes. valid answer.		yes. valid answer.

$x\in\{-9,4\}$

We use the "interval method" because not only will simple equations, such as this, work nicely but also far more complicated equations will work perfectly using this same method.

We replace example 5 in the text with:



$|2x-3|-x+1=|x+5|$

The intervals are $x<-5$ $\quad -5 \le x < \frac{3}{2}$ $\quad$ and $x \ge \frac{3}{2}$

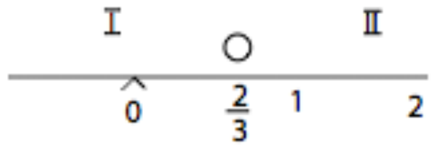
If $x<-5$	If $-5 \le x < \frac{3}{2}$	If $x \ge \frac{3}{2}$
$-(2x-3)-x+1=-(x+5)$	$-(2x-3)-x+1=x+5$	$(2x-3)-x+1=x+5$
$-2x+3-x+1=-x-5$	$-2x+3-x+1=x+5$	$2x-3-x+1=x+5$
$-3x+4=-x-5$	$-3x+4=x+5$	$x-2=x+5$
$9=2x$	$-1=4x$	$-2=5$
$\frac{9}{2}=x$	$-\frac{1}{4}=x$	
Reject. not in interval	valid	Reject. False.

the only solution to this problem is $x=-\frac{1}{4}$

Notice that **only** those "zeros" of the insides of the absolute values were marked on our number line.

This original equation could have been written as: $|2x-3|+(1-x)=|x+2|$ and its solution would have been exactly the same as we did here.

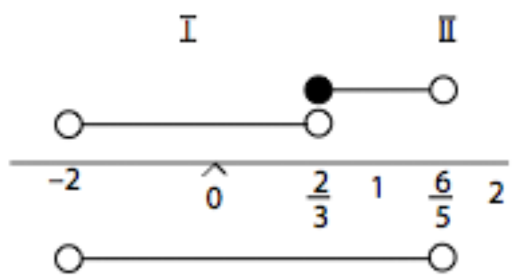
The replacement for example 8: $|3x-2| < 4-2x$

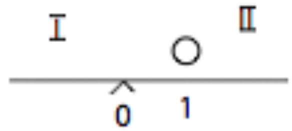


Our two intervals are $x < \frac{2}{3}$ and $x \geq \frac{2}{3}$

If $x < \frac{2}{3}$	If $x \geq \frac{2}{3}$
$-(3x-2) < 4-2x$	$(3x-2) < 4-2x$
$-3x+2 < 4-2x$	$3x-2 < 4-2x$
$-2 < x$	$5x < 6$
	$x < \frac{6}{5}$
is -2 in interval?	is $\frac{6}{5}$ in interval?
Yes, so	Yes, so
$-2 < x < \frac{2}{3}$	$\frac{2}{3} \leq x < \frac{6}{5}$

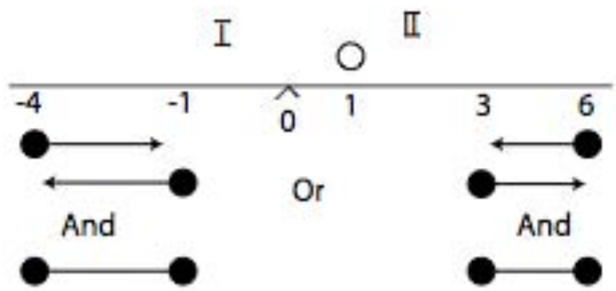
The conclusion is $-2 < x < \frac{6}{5}$





Example: $2 \le |x-1| \le 5$

If $x<1$	If $x \ge 1$
$2 \le -(x-1) \le 5$	$2 \le x-1 \le 5$
$2 \le -x+1 \le 5$	$3 \le x \le 6$
$1 \le -x \le 4$	
$-1 \ge x \ge -4$	



The solution is written as one of these:

$$x \in \{[-4,-1] or [3,6]\}$$

$$[-4,-1] \cup [3,6]$$

The square brackets are used because of the equality.

These problems are to be done ***after*** the class lecture for section 4.3.

For each problem, make a number line showing zero. Indicate zeros of any absolute value expressions. Label intervals using roman numerals and write the specific interval being used in proper format.

For each interval, properly remove the absolute value and solve the resulting equation. Determine the suitability of the solution and finally, draw another number line showing the final solution and write the algebraic expression for that/those interval(s).

A) $|x+1|+|x-2| \leq 7$

B) $|2x+1|+|2-x| \leq 7$

C) $|x+1|-|2x-1| \leq -4$

D) $|2x|-|x+2| \geq 3$

E) $|x+2| < |x-3|$

F) $|x+2|+2x \geq |x-3|$

Algebraic solutions on next page.

A) $-3 \le x \le 4$

B) $-2 \le x \le \frac{8}{3}$

C) $x \le -2$ or $x \ge 6$

D) $x \le -\frac{5}{3}$ or $x \ge 5$

E) $x < \frac{1}{2}$

F) $x \ge \frac{1}{4}$